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On flows in bidirected graphs

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Abstract

Bouchet conjectured that every bidirected graph which admits a nowhere-zero bidirected flow will admit a nowhere-zero bidirected 6-flow [A. Bouchet, Nowhere-zero integer flows on a bidirected graph, J. Combin. Theory Ser. B 34 (1983) 279–292]. He proved that this conjecture is true with 6 replaced by 216. Zyka proved in his Ph.D dissertation that it is true with 6 replaced by 30. Khelladi proved it is true with 6 replaced with 18 for 4-connected graphs [A. Khelladi, Nowhere-zero integer chains and flows in bidirected graphs, J. Combin. Theory Ser. B 43 (1987) 95–115]. In this paper, we prove that Bouchet's conjecture is true for 6-edge connected bidirected graphs.

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1. Introduction

A bidirected graph G is a directed graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities: $\bullet \rightarrow \bullet \leftarrow \bullet$, $\bullet \leftarrow \bullet \rightarrow \bullet$, $\bullet \rightarrow \bullet \rightarrow \bullet$, $\bullet \leftarrow \bullet \leftarrow \bullet$. An edge with orientation $\bullet \rightarrow \bullet \leftarrow \bullet$ (resp., $\bullet \leftarrow \bullet \rightarrow \bullet$) is called an *in-edge* (resp., *out-edge*). Edges with other orientations are called *ordinary edges*. The set of in(out)-edges is denoted by $E_i(G)$ ($E_o(G)$, respectively). We define $O(G) = |E_i(G)| + |E_o(G)|$.

For an ordinary edge e , reversing the orientation of e means the natural way to change its orientation. For any non-ordinary edge $e \in E(G)$, reversing the orientation of e means

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changing e from an in(out)-edge to an out(in)-edge. Note that after reversing the orientation of an edge of G , $O(G)$ remains the same.

Let G be a bidirected graph. For any $v \in V(G)$, the set of all edges with tails (or heads) at v is denoted by $E^+(v)$ (or $E^-(v)$) and we define $E(v) = E^+(v) \cup E^-(v)$. (For a general graph G , we use $E(v)$ to denote the set of edges which are incident with v in G .) A bidirected graph is eulerian if $|E(v)|$ is even for each $v \in V(G)$. Readers are referred to [1] for terminology not defined in this paper.

Definition 1.1. Let G be a bidirected graph and f be a function: $E(G) \mapsto \mathbb{Z}$. Then:

- (1) f is called a *bidirected k -flow* of G if $-k + 1 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$;
- (2) f is called a *bidirected modular k -flow* of G if $0 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{k}$ for every $v \in V(G)$;
- (3) The *support* of a bidirected k -flow (resp., modular k -flow) f of G is the set of edges of G with $f(e) \neq 0$ (resp., $f(e) \not\equiv 0 \pmod{k}$), and is denoted by $\text{supp}(f)$. A bidirected k -flow (modular k -flow) of G is *nowhere-zero* if $\text{supp}(f) = E(G)$.

For nowhere-zero bidirected integer flows, Bouchet [2] proposed the following conjecture (see also Toft and Jensen's book [11]).

Conjecture 1.2. Every bidirected graph which admits a nowhere-zero bidirected flow will admit a nowhere-zero bidirected 6-flow.

Note that the value 6 in Conjecture 1.2 is best possible, see [2] for details. The following is a list of the partial results to Conjecture 1.2.

Theorem 1.3. Let G be a bidirected graph admitting a nowhere-zero bidirected flow. Then:

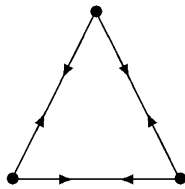
- (1) (Bouchet [2]) G admits a nowhere-zero bidirected 216-flow;
- (2) (Zyka [16], or see [6]) G admits a nowhere-zero bidirected 30-flow;
- (3) (Khelladi [6]) G admits a nowhere-zero bidirected 18-flow if G is 4-connected.

Bidirected flow is a generalization of the concept of integer flow (introduced by Tutte [12,15] as a dual version for the vertex coloring problem). This is because a directed graph G can be considered as a bidirected graph G^* with $O(G^*) = 0$. However, bidirected flows and integer flows can be quite different due to the existence of the in-edges and out-edges. Some results for integer flows can be generalized to bidirected flows while some other results cannot. The following observation for bidirected 2-flows is a generalization of Tutte's 2-flow characterization. Though the proof for integer 2-flows is straightforward, the corresponding result for bidirected 2-flow does need a few more steps in the proof.

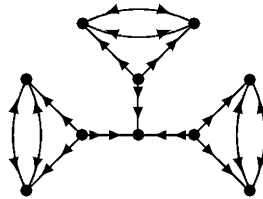
Proposition 1.4. Let G be a connected bidirected graph. Then, G admits a nowhere-zero bidirected 2-flow if and only if G is a connected eulerian graph and $O(G)$ is even.

We will provide a proof of this proposition in Section 2.

Integer flows and modular flows are proved to be equivalent for general graphs (see [12,13] or [15]). However, for bidirected graphs, they are not equivalent to each other in many cases. The following are some examples that a nowhere-zero modular 2-flow ($f(e) = 1$ for each $e \in E(G)$) in Example 1 and a nowhere-zero modular 3-flow in Example 2 ($f(e) = 1$ for each $e \in E(G)$) cannot be converted to a nowhere-zero bidirected 2-flow (or 3-flow). The fact that the graph in Example 2 has no bidirected 3-flow will follow from Lemma 3.2. For nowhere-zero bidirected modular 3-flows, we are able to establish their equivalent relation for 2-edge connected bidirected graphs.



Example 1



Example 2

The following is one of our main results in this paper which provides a major step in the proof of Theorem 1.6.

Theorem 1.5. *Let G be a 2-edge connected bidirected graph. Then G admits a nowhere-zero bidirected 3-flow if and only if G admits a nowhere-zero modular bidirected 3-flow.*

By applying Theorem 1.5, we are able to verify Conjecture 1.2 for 6-edge connected bidirected graphs.

Theorem 1.6. *Let G be a 6-edge connected bidirected graph. Then G admits a nowhere-zero bidirected flow if and only if it admits a nowhere-zero bidirected 6-flow.*

2. Proof of Proposition 1.4 and its applications

A *circuit* is a connected 2-regular graph and a *cycle* is a graph such that every vertex is of even degree. Let G be a bidirected graph. A circuit C of G is said to be *balanced* provided that $O(C)$ is even and *unbalanced* otherwise. A cycle P of G is said to be *balanced* provided that for each of its components P' , $O(P')$ is even. A collection of cycles $F = \{P_1, P_2, \dots, P_r\}$ of G is called a *proper r -cycle cover* of G if every P_i is balanced and $\bigcup_{i=1}^r E(P_i) = E(G)$.

Proof of Proposition 1.4. “ \implies ”. Note that if f is a nowhere-zero bidirected 2-flow of G and we reverse the orientation of some edge e_1 , then the resulting graph G_1 admits a nowhere-zero bidirected 2-flow f_1 , such that, $f_1(e) = f(e)$ for any $e \neq e_1$ and $f_1(e_1) = -f(e_1)$. Also, $O(G) = O(G_1)$.

So we may assume that G admits a nowhere-zero bidirected 2-flow f such that $f(e) = 1$ for every edge $e \in E(G)$. Therefore, for any $v \in V(G)$, $E^+(v) = E^-(v)$ which implies

G is an eulerian graph. Since $\sum_{v \in V(G)} E^+(v) = \sum_{v \in V(G)} E^-(v)$ and each ordinary edge contributes 1 to both $\sum_{v \in V(G)} E^+(v)$ and $\sum_{v \in V(G)} E^-(v)$, the total contribution of non-ordinary edges to $\sum_{v \in V(G)} E^+(v)$ must be the same as that to $\sum_{v \in V(G)} E^-(v)$. Therefore, we get $|E_i(G)| = |E_o(G)|$ and consequently, $O(G)$ is even.

“ \Leftarrow ”. Since G is an eulerian graph, there is a circuit decomposition $F = \{C_1, C_2, \dots, C_t\}$ of $E(G)$. We will prove it by induction on t .

Suppose that $t = 1$, then G is a circuit. Since $O(G)$ is even, let $\{e_1, e_2, \dots, e_{2m}\}$ be the set of non-ordinary edges of G such that e_1, e_2, \dots, e_{2m} appear in the circuit in this order. Clearly, if we reverse the orientation of some edges in G to get another graph G' , then G admits a nowhere-zero bidirected l -flow if and only if G' does.

Let us reverse the orientation of some non-ordinary edges in G so that e_i is an in-edge if i is odd and an out-edge otherwise. Because there are even number of non-ordinary edges, every two consecutive (in the circuit order) non-ordinary edges are of different types, i.e., one is an in-edge and the other one is an out-edge. Now, it is easy to reverse the orientation of some ordinary edges so that for the resulting graph G' , $|E^+(v)| = |E^-(v)| = 1$ for any $v \in V(G')$. Then, G' admits a nowhere-zero bidirected 2-flow with $f(e) = 1$ for each $e \in E(G)$. It follows that G admits a nowhere-zero bidirected 2-flow.

For $t \geq 2$, since G is connected, then there exists C_i such that the induced subgraph $G^* = G[E(G) \setminus E(C_i)]$ is connected. Note that $O(C_i)$ is even if and only if $O(G^*)$ is even.

Assume that $O(C_i)$ is even. By induction hypothesis, both C_i and G^* admits nowhere-zero bidirected 2-flows, therefore G admits a nowhere-zero bidirected 2-flow. So, we may assume that $O(C_i)$ is odd ($O(G^*)$ is odd as well). Let v be any vertex in $V(C_i) \cap V(G^*)$. Suppose $v \in C_j$ for some $j \neq i$. Let C'_i be the new circuit obtained from C_i by splitting an edge $e_i = vv_i$ away from v (becomes $e_i = v'_i v_i$, where v'_i is a new vertex) and adding a new in-edge between v and v'_i (this operation is called “expending v in C_i to an in-edge $e_i = vv'_i$ ”). Similarly, we get a new graph G' from G^* by expending v in C_j to an out-edge $e_j = vv'_j$. Clearly, both $O(C'_i)$ and $O(G')$ are even. By induction hypothesis, C'_i admits a nowhere-zero bidirected 2-flow f_1 and G' admits a nowhere-zero bidirected 2-flow f_2 . Note that $-f_1$ is also a nowhere-zero bidirected 2-flow of C'_i . So, we can assume $f_1(e_i) = f_2(e_j)$ (otherwise since $-f_1(e_i) = f_2(e_j)$ we will use $-f_1$ instead of f_1). Therefore, let us define

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(C_i), \\ f_2(e) & \text{if } e \in E(G) \setminus E(C_i). \end{cases}$$

Clearly, f is a nowhere-zero bidirected 2-flow of G . \square

The following result is a generalization of a theorem about integer flows.

Proposition 2.1. *Let G be a bidirected graph. If G has a proper r -cycle cover, then G admits a nowhere-zero bidirected 2^r -flow.*

Proof. Let $F = \{P_1, P_2, \dots, P_r\}$ be a proper r -cycle cover of G . By Proposition 1.4, each P_i admits a nowhere-zero bidirected 2-flow f_i . It is easy to verify that $\sum_{i=1}^r 2^{r-1} f_i$ is a nowhere-zero bidirected 2^r -flow. \square

3. Proof of Theorem 1.5

Let us first introduce a useful lemma whose proof is straightforward.

Lemma 3.1. *Let G be a bidirected graph, E_0 be a subset of E and G_{E_0} be the bidirected graph obtained from G by reversing the orientation of each edge in E_0 . Then we have:*

- (1) *G admits a bidirected k -flow if and only if G_{E_0} admits a bidirected k -flow with the same support.*
- (2) *G admits a bidirected modular k -flow if and only if G_{E_0} admits a bidirected modular k -flow with the same support.*

Proof. (1) Suppose that f is a bidirected k -flow of G . Let

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin E_0, \\ -f(e) & \text{if } e \in E_0. \end{cases}$$

Clearly, f' is a bidirected k -flow of G_{E_0} with the same support.

(2) Suppose that f is a bidirected modular k -flow of G . Let

$$f'(e) = \begin{cases} k - f(e) & \text{if } e \in E_0 \text{ and } f(e) \neq 0, \\ f(e) & \text{otherwise.} \end{cases}$$

Clearly, f' is a bidirected modular k -flow of G_{E_0} with the same support. \square

Lemma 3.2. *Let G be a bidirected cubic graph. Then G admits a nowhere-zero bidirected 3-flow if and only if G admits a nowhere-zero bidirected modular 3-flow and G has a perfect matching.*

Proof. “ \Rightarrow ”. Let f be a nowhere-zero bidirected 3-flow of G and $E_0 = \{e \mid f(e) = -1 \text{ or } -2\}$. By the proof of Lemma 3.1(1), G_{E_0} admits a positive bidirected 3-flow f' . Clearly, f' is also a nowhere-zero bidirected modular 3-flow of G_{E_0} . By Lemma 3.1(2), G admits a nowhere-zero bidirected modular 3-flow. Because for every vertex $v \in V(G)$, there is exactly one incident edge e such that $f'(e) = 2$, then $E' = \{e \mid f'(e) = 2\}$ is a perfect matching of G .

“ \Leftarrow ”. Let f be a nowhere-zero bidirected modular 3-flow of G and $E_0 = \{e \mid f(e) = 2\}$. By Lemma 3.1(2), $G^* = G_{E_0}$ admits a nowhere-zero bidirected modular 3-flow f' such that $f'(e) = 1$ for each edge e . Since G^* is a cubic graph, then for any vertex v either $E(v) = E^+(v)$ or $E(v) = E^-(v)$. Let M be a perfect matching of G^* . Then reverse the direction of each edge e of M and change the value $f'(e)$ to be 2. The resulting nowhere-zero bidirected modular 3-flow is also a nowhere-zero bidirected 3-flow of G_M^* . By Lemma 3.1(1), G^* , and therefore G admit nowhere-zero bidirected 3-flows. \square

Definition 3.3. Let G be a graph and v be a vertex of G . Suppose that $F \subset E(v)$, then we use $G_{[v;F]}$ for the graph obtained from G by splitting the edges of F away from v , that is, adding a new vertex v' and changing the end v of the edges in F to be v' .

Lemma 3.4 (Nash-Williams [8]). *Let k be an even integer and G be a k -edge connected graph and $v \in V(G)$. Let a be an integer such that $k \leq a$ and $k \leq d_G(v) - a$. Then there is an edge set $F \subset E(v)$ such that $|F| = a$ and $G_{[v;F]}$ is k -edge connected.*

Lemma 3.5 (Fleischner [3]). *Let G be a 2-edge connected graph. Suppose that v is a vertex of G with $d_G(v) \geq 4$ and let $e_0, e_1, e_2 \in E(v)$. Then either $G_{[v;\{e_0, e_1\}]}$ or $G_{[v;\{e_0, e_2\}]}$ is 2-edge connected or $G_{[v;\{e_0, e_1, e_2\}]}$ has more components than G .*

Lemma 3.6. *Let G be a 2-edge connected graph. Suppose that v is vertex of G with $d_G(v) \geq 4$ and let $e_0, e_1, e_2, e_3 \in E(v)$. Then*

- (1) *at least one of $G_{[v;\{e_0, e_i\}]}$ ($i = 1, 2, 3$) is 2-edge connected;*
- (2) *in the case of $d_G(v) = 4$, at least one of $G_{[v;\{e_0, e_i\}]}$ ($i = 1, 2$) is 2-edge connected.*

Proof. By Lemma 3.5, it suffices to show that at least one of $G_{[v;\{e_0, e_1, e_2\}]}$ and $G_{[v;\{e_0, e_1, e_3\}]}$ is connected. For convenience, let $e_i = vv_i$ ($i = 0, 1, 2, 3$). If $d_G(v) = 4$, then $G_{[v;\{e_0, e_1, e_2\}]}$ is connected. Otherwise e_3 will be an edge cut of G which is impossible. So, we may assume that $d_G(v) \geq 5$. Suppose that $G_{[v;\{e_0, e_1, e_2\}]}$ is disconnected. Since G is 2-edge connected then there is a path P in $G_{[v;\{e_0, e_1, e_2\}]}$ which connects v_2 to v_0 without using e_2 . Therefore, $G_{[v;\{e_0, e_1, e_3\}]}$ is connected. \square

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. “ \Rightarrow ”. Suppose that G admits a nowhere-zero bidirected 3-flow. Using the similar argument in the proof of the first part of Lemma 3.2, we can get a nowhere-zero bidirected modular 3-flow of G .

“ \Leftarrow ”. Suppose that it is not true. Let G be a smallest counterexample with respect to $|E(G)|$. By Lemma 3.1, we may assume that G admits a nowhere-zero bidirected modular 3-flow f such that $f(e) = 1$ for each edge e . Then for any vertex v with $d_G(v) = 3$ we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

Claim 1. $\delta(G) \geq 3$.

Otherwise, suppose that there exists $v \in V(G)$ such that $N_G(v) = \{v_1, v_2\}$. Since f is a nowhere-zero bidirected modular 3-flow of G and $f(v_1v) = f(vv_2) = 1$, we can delete the edges v_1v, vv_2 from G and add a new edge v_1v_2 (or a parallel edge if v_1v_2 exists). If $v_1v \in E^+(v_1)$ ($E^-(v_1)$), then orient this new edge such that this $v_1v_2 \in E^+(v_1)$ ($E^-(v_1)$); if $v_2v \in E^+(v_2)$ ($E^-(v_2)$), then orient this new edge such that this $v_1v_2 \in E^+(v_2)$ ($E^-(v_2)$). The resulting bidirected graph is denoted by G_0 . Then G_0 is 2-edge connected and admits a nowhere-zero bidirected modular 3-flow and $|E(G_0)| < |E(G)|$. By the choice of G , G_0 admits a nowhere-zero bidirected 3-flow. Clearly, from this flow we can easily get a nowhere-zero bidirected 3-flow of G , a contradiction.

Claim 2. *There is no $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$.*

Assume that there exists a vertex $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$. Then $d_G(v) \geq 4$, since, for any degree 3 vertex, either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

If $|E^-(v)|$ or $|E^+(v)| \geq 3$, then we may apply Lemma 3.6(1) to split two edges e' , e'' away from the vertex v where $e' \in E^+(v)$ and $e'' \in E^-(v)$, and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Clearly, this new graph has a degree two vertex and if this graph admits a nowhere-zero 3-flow, then so does G . Similar to the proof of Claim 1, we can get a contradiction.

If $0 < |E^-(v)|, |E^+(v)| \leq 2$, then $d_G(v) = 4$ since $|E^-(v)| \equiv |E^+(v)| \pmod{3}$. By Lemma 3.6(2) and with the same argument as in Case 1, we get a contradiction as well.

By Claims 1 and 2, we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$ for any $v \in V(G)$. Therefore, $d_G(v) \equiv 0 \pmod{3}$ for any $v \in V(G)$. For any v with $d_G(v) \geq 6$, we may apply Lemma 3.4 to split this vertex into several degree 3 vertices and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Also, if the resulting graph admits a nowhere-zero bidirected 3-flow, then so does G . Recursively splitting high-degree vertices, we obtain a 2-edge connected cubic graph G^* which admits a nowhere-zero bidirected modular 3-flow.

By Petersen Theorem [9], the 2-connected, cubic graph G^* has a perfect matching. Then by Lemma 3.2, G^* admits a nowhere-zero bidirected 3-flow. So, from this flow of G^* , we can get a nowhere-zero bidirected 3-flow of G , a contradiction. \square

4. Proof of Theorem 1.6

Product of flows is a method used in the proof of the theorems of 8-flow [4], 4-flow [5] and 6-flow [10]. The following lemma generalizes this method for bidirected flows of graphs and is used in the proof of Theorem 1.6.

Lemma 4.1. *Let G be a bidirected graph. If G admits a bidirected k_1 -flow f_1 and a bidirected k_2 -flow f_2 such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$, then G admits a nowhere-zero bidirected $k_1 k_2$ -flow.*

Proof. Let $f(e) = f_1(e) + k_1 f_2(e)$ for every $e \in E(G)$, then it is easy to verify that f is a nowhere-zero bidirected $k_1 k_2$ -flow. \square

Let G be a bidirected graph. A subgraph C of G is called a *bidirected circuit* of G provided that either C is a balanced circuit of G or C is the union of two unbalanced circuits sharing exactly one vertex or C is the union of two vertex-disjoint unbalanced circuits and a simple path meeting each of the two circuits at exactly one of its end points.

We extend Seymour's closure operation [10] to bidirected graphs as follows:

For a positive integer k , if X is a subgraph of G , then the k -closure of X in G , denoted by $\langle X \rangle_k$, is the (unique) maximal subgraph of G of the form $X \cup C_1 \cup \dots \cup C_n$, where for every i , $1 \leq i \leq n$, C_i is a bidirected circuit of G and $|E(C_i) \setminus E(X \cup C_1 \cup \dots \cup C_{i-1})| \leq k$.

By [6, Theorem 4.3], we have the following:

Lemma 4.2. *Let G be a connected bidirected graph which admits a nowhere-zero bidirected flow. Let E' be a subset of $E(G)$ such that the induced graph $G[E']$ is connected and spanning. Then, $\langle E' \rangle_2 = E(G)$.*

By [6, Proposition 2.2], we have the following:

Lemma 4.3. *Let G be a bidirected graph and $k \geq 3$ be a prime integer. Let E' be a subset of $E(G)$ such that $\langle E' \rangle_{k-1} = E(G)$. Then G admits a bidirected integer flow f , such that, $f(e) \not\equiv 0 \pmod{k}$ for every element e of $E(G) \setminus E'$.*

The bidirected flow f obtained in Lemma 4.3 can be considered as a bidirected modular k -flow with the same support, though the absolute value of $f(e)$ can be very large for $e \in E(G) \setminus E'$.

We will use the following lemma to get some edge-disjoint spanning trees.

Lemma 4.4 (Nash-Williams [7], Tutte [14]). *Every $2k$ -edge-connected graph contains k -edge-disjoint spanning trees.*

Definition 4.5. Let H_1 and H_2 be two subgraphs of a graph G . The *symmetric difference* of H_1 and H_2 , denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$. The symmetric difference of finitely many subgraphs $\{H_1, \dots, H_t\}$ of G is defined recursively as

$$\triangle_{1 \leq i \leq t} H_i = H_1 \triangle \dots \triangle H_{t-1} \triangle H_t = [H_1 \triangle \dots \triangle H_{t-1}] \triangle H_t.$$

The following is a well-known fact:

Lemma 4.6. *Let $\{H_1, \dots, H_t\}$ be a family of subgraphs of G . Let $S = H_1 \triangle \dots \triangle H_t$. Then:*

- (1) S is the subgraph of G induced by the edges contained in an odd number of H_i 's;
- (2) If H_1, \dots, H_t are all cycles of G , then S is also a cycle.

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. “ \Rightarrow ”. Clearly, a nowhere-zero bidirected 6-flow is also a nowhere-zero bidirected flow.

“ \Leftarrow ”. Since G is 6-edge connected, by Lemma 4.4, G has three edge-disjoint spanning trees T_1, T_2 and T_3 . Because G admits a nowhere-zero bidirected flow, by Lemma 4.2, $\langle T_i \rangle_2 = E(G)$ for $1 \leq i \leq 3$.

For $1 \leq i \leq 3$, let $k = 3$, $E' = E(T_i)$ and apply Lemma 4.3, we get a bidirected integer flow f_i of G such that $f_i(e) \not\equiv 0 \pmod{3}$ for every element e of $E(G) \setminus E(T_i)$. Clearly, $\text{supp}(f_i) \supseteq E(G) \setminus E(T_i)$. Since $G[\text{supp}(f_i)]$ contains two edge-disjoint spanning trees T_2 and T_3 , the subgraph $G[\text{supp}(f_i)]$ is 2-edge connected and spanning. Similarly, $\text{supp}(f_2) \supseteq E(G) \setminus E(T_2)$, $\text{supp}(f_3) \supseteq E(G) \setminus E(T_3)$ and both $G[\text{supp}(f_2)]$ and $G[\text{supp}(f_3)]$ are 2-edge connected and spanning. From each f_i ($1 \leq i \leq 3$), we can get a bidirected modular 3-flow

f'_i such that $\text{supp}(f_i) = \text{supp}(f'_i)$. Then, by Theorem 1.5, we get a bidirected 3-flow f''_i such that $\text{supp}(f''_i) = \text{supp}(f_i)$ for $i = 1, 2, 3$.

Let T_i, T_j be two edge-disjoint spanning trees. For each $e \in T_j$, there is a unique circuit C_e of G contained in $T_i \cup \{e\}$. By Lemma 4.6(2), $C_i^j = \triangle_{e \in E(T_j)} C_e$ is a cycle. By Lemma 4.6(1), $(E(T_i) \cup E(T_j)) \supseteq E(C_i^j) \supseteq E(T_j)$, then C_i^j is also connected and spanning. Let us consider the following two cases:

Case 1: There exists one C_i^j such that $O(C_i^j)$ is even.

Without loss of generality, we may assume that $O(C_1^2)$ is even. Since C_1^2 is eulerian and connected, by Proposition 1.4, C_1^2 admits a nowhere-zero bidirected 2-flow. Therefore, G admits a bidirected 2-flow $f_{1,2}$ such that $\text{supp}(f_{1,2}) \supseteq E(T_2)$. Also f_2'' is bidirected 3-flow such that $\text{supp}(f_2'') \supseteq E(G) \setminus E(T_2)$. By Lemma 4.1, we get a nowhere-zero bidirected 6-flow of G .

Case 2: All $O(C_i^j)$'s are odd for $1 \leq i \neq j \leq 3$.

Since $O(C_1^3)$ is odd, the eulerian subgraph C_1^3 contains an unbalanced circuit c_1^3 . Clearly, $E(c_1^3) \subseteq (E(T_1) \cup E(T_3))$. Because $O(C_1^2)$ is odd, $E(T_2) \subseteq E(C_1^2) \subseteq E(T_1) \cup E(T_2)$ and $E(T_2) \subseteq E(C^*)$, then $C^* = C_1^2 \triangle c_1^3$ is a connected spanning cycle of G with $O(C^*)$ even. By Proposition 1.4, C^* admits a nowhere-zero bidirected 2-flow f^* with $\text{supp}(f^*) \supseteq E(T_2)$. Similar to Case 1, by Lemma 4.1, we can get a nowhere-zero bidirected 6-flow of G . \square

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